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ON THE SEMI-CONTINUITY OF MINIMAL LOG DISCREPANCIES

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ABSTRACT. We show that the minimal log discrepancy function of 3-folds and toric varieties is lower semi-continuous. This semi-continuity property strenghtens the sharp upper bound of minimal log discrepancies conjectured by V.V. Shokurov.

0. INTRODUCTION

The Minimal Model Program predicts that an algebraic variety can be transformed into a minimal model after a finite sequence of surgery operations (divisorial contractions and flips). Singularities appear naturally in the process, and it is expected that varieties with *only log canonical singularites* form the largest class in which the Minimal Model Program works.

To any valuation centered on a a variety X with only log canonical singularities, one can associate a non-negative rational number, called *log discrepancy*. The *minimal log discrepancy* of X in a given point is the minimum of log discrepancies of all valuations centered at that point [Sh88]. For instance, the minimal log discrepancy $a(x; X)$ of a variety X in a nonsingular point $x \in X$ coincides with the log discrepancy of the exceptional locus of the blow-up centered at x, and equals the codimension of x.

Discrepancies have been used for instance to obtain effective bounds for the generation of adjoint line bundles (cf.[ELM95, Ka97]). They are related to the factorization of birational maps between Mori fibre spaces (cf.[Co99]), and also to the classification of divisorial contractions (cf.[Ka94, Co99, K1, K2]).

A basic property of discrepancies is that they increase after each basic step of the Minimal Model Program, thus the termination of the Minimal Model Program process in a finite number of steps seems to rely on certain spectral properties of discrepancies. V.V. Shokurov proposed the following A.C.C. Conjecture, which is proven in codimension

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two [Al93, Sh91], and for $\Gamma = \{0\}$ in the case of toric varieties [Br97] (see Section 1 for definitions and notations):

Conjecture 0.1. [Sh88] *If $\Gamma \subset [0,1]$ is a subset satisfying the descending chain condition, then the set*

$$A(\Gamma, n) := \{a(\eta; B); (X, B) \textit{ log variety}, \operatorname{codim}(\eta, X) = n, b_j \in \Gamma\ \forall j\}$$

satisfies the ascending chain condition.

An equivalent statement is that $A(\Gamma, n)$ is bounded from above and has no accumulation points from below. The following conjecture, proven up to codimension three [Mrk96, Ka93] based on the classification of terminal 3-fold singularities [Rd80, Mr85], proposes a sharp upper bound:

Conjecture 0.2. [Sh88] *Let (X, B) be a log variety and let $\eta \in X$ be a Grothendieck point. Then the following inequality holds:*

$$a(\eta; B) \leq \operatorname{codim} \eta.$$

Moreover, X is nonsingular in η if $a(\eta; B) > \operatorname{codim} \eta - 1$.

Our goal is to strengthen the first part of Conjecture 0.2:

Main Theorem 1. *Let (X, B) be a log variety, and let x be a closed point on a curve C in X. Assume one of the following extra assumptions is satisfied:*

a) $\dim X \leq 3$, *or*

b) *X is a torus embedding and B is invariant under the torus action.*

Then $a(x; B) \leq a(\eta_C; B) + 1$.

We will show that the conclusion of the Main Theorem is equivalent to the lower semi-continuity of minimal log discrepancies of the points on a given log variety. Our result does not prove any new case of Conjecture 0.2, but we hope that the lower semi-continuity is the reason behind the conjectured sharp upper bound. We remark that the somehow related upper semi-continuity of thresholds in a family of hypersurfaces has been proved by analytic methods (cf.[DK00, PS00]).

This note is based on [Am99], which we refer to for futher details.

1. Preliminary

A *variety* is a reduced irreducible scheme of finite type over a fixed field k, of characteristic 0. An *extraction* is a proper birational morphism of normal varieties. A *log pair* (X, B) is a normal variety X equipped with an $\mathbb{R}$-Weil divisor B such that $K + B$ is $\mathbb{R}$-Cartier. (X, B) is called a *log variety* if moreover, B is effective.

If (X, B) is a log pair and $\mu : \tilde{X} \to X$ is an extraction, the *log codiscrepancy divisor* of (X, B) on $\tilde{X}$ is the unique divisor $\tilde{B}$ on $\tilde{X}$ such that $\mu^*(K+B) = K_{\tilde{X}} + \tilde{B}$ and $\tilde{B} = \mu^{-1}B$ on $\tilde{X} \backslash Exc(\mu)$. The identity $\tilde{B} = \sum_{E \subset \tilde{X}} (1 - a(E; X, B))E$ associates to each prime divisor E of $\tilde{X}$ a real number $a(E; X, B)$, called the *log discrepancy* of E with respect to (X, B). The invariant $a(E; X, B)$ depends only on the valuation with center $c_X(E) = \mu(E)$ defined by E on the field of rational functions of X. For simplicity, we may write $a(E; B)$ for $a(E; X, B)$.

Definition 1.1. [Sh88] The *minimal log discrepancy* of a log pair (X, B) at a proper Grothendieck point $\eta \in X$ is defined as

$$a(\eta; X, B) = \inf_{c_X(E)=\eta} a(E; X, B),$$

where the infimum is taken after all prime divisors on extractions of X having η as a center on X. We set by definition $a(\eta_X; X, B) = 0$.

The log pair (X, B) has only *log canonical singularities* if $a(\eta; B) \geq 0$ for every proper point $\eta \in X$.

2. Lower semi-continuity

Definition 2.1. Let (X, B) be a log pair. The *mld-spectrum* of (X, B) is defined as the set $\mathrm{Mld}(X, B) := \{a(\eta; B); \eta \in X\} \subset \{-\infty\} \cup \mathbb{R}$. We denote by a^o the map $X \to \mathrm{Mld}(X; B)$ $(x \mapsto a(x; B))$, defined on the closed points of X. The partition of X into the fibers of the map a^o is called the *mld-stratification* of (X, B).

Lemma 2.2. *Assume $W \subset X$ is a closed irreducible subvariety and (X, B) is a log pair with only log canonical singularities at η_W. Then there exists an open subset U of X such that $U \cap W \neq \emptyset$ and*

$$a(x; B) = a(\eta_W; B) + \dim W$$

for every closed point $x \in W \cap U$.

Theorem 2.3. *Given a log pair (X, B), the mld-spectrum* $\mathrm{Mld}(X, B)$ *is a finite set, and the mld-stratification is constructible, i.e. all the fibers of the map a^o are constructible sets.*

Proposition 2.4. *For a log variety (X, B), the following statements are equivalent:*

1. *The function a^o is lower semi-continuous. That is, every closed point $x \in X$ has a neighborhood $x \in U \subseteq X$ such that $a(x; B) = \inf_{x' \in U} a(x'; B)$.*
2. *$a(x; B) \leq a(\eta_C; B) + 1$ for every closed point x on a curve C in X.*

Proof. Assume (2) holds, and let $x \in X$ be a closed point. By Theorem 2.3, we may shrink X such that $x \in \bar{C}$ for every irreducible component C of the fibers of the map a^o. For $x' \in X$, there exists a C such that $x' \in C$. Since $x \in \bar{C}$, we infer that $a(x; B) \leq a(\eta_C; B) + \dim \eta_C = a(x'; B)$.

Assume (1) holds. Let U_x be a neighborhood of x such that $a(x; B) \leq a(x'; B)$ for all $x' \in U_x$. Then $U_x \cap \bar{\xi} \subset \bar{\xi}$ is an open dense subset. From Lemma 2.2, there exists some $x' \in U_x \cap \bar{\xi}$ such that $a(x'; B) = a(\xi; B) + \dim \xi$. Therefore $a(x; B) \leq a(\xi; B) + \dim \xi$. □

Proof of the Main Theorem. (a) If X is a curve or surface, the desired inequality is easily checked, so let us assume X is a 3-fold. We may assume $a(x; B) > 1$, thus (X, B) has only log canonical singularities.

Assume first $a(\eta_C; B) \leq 1$. From the Log Minimal Model Program (cf. [Ka92]), there exists a crepant extraction $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ such that $\tilde{B}$ is effective and there exists a prime divisor E on $\tilde{X}$ with $\mu(E) = C$ and $a(\eta_E; \tilde{B}) = a(\eta_C; B)$. Let η be the generic point of a curve in the fiber of $\mu|_E : E \to C$ over x. Then, $a(x; B) \leq a(\eta; \tilde{B}) \leq a(\eta_E; \tilde{B}) + 1$, where the latter inequality holds by the 2-dimensional case.

Assume now $a(\eta_C; B) > 1$. Then we may assume $a(x; B) > 2$, thus X is nonsingular at both x and η_C and $a(x; B) - (a(\eta_C; B) + 1) = \operatorname{mult}_C B - \operatorname{mult}_x B \leq 0$.

(b) Assume $X = T_N(\Delta)$ is toric variety and $B = \sum_i (1 - a_i) B_i$ is an invariant divisor. By definition, there exists a piecewise linear form φ on $N_{\mathbb{R}}$ such that $\varphi(v_i) = a_i$ for every i, where $\{v_i\}$ are the primitive vectors on the rays of Δ. We may assume the log variety (X, B) has only log canonical singularities, i.e. $0 \leq a_i \leq 1$ for every i. Then we have the following formula for the minimal log discrepancies of (X, B) at the generic points of the orbits:

$$a_\sigma := a(\eta_{\mathrm{orb}(\sigma)}; B) = \inf\{\varphi(v); v \in \text{rel int}(\sigma) \cap N\}, \ \sigma \in \Delta.$$

First of all, each strata in the mld-stratification is a union of orbits of the torus action. This follows from Lemma 2.2 and the transitivity of the torus action. Then it is enough to check that $a_\sigma + \operatorname{codim}(\sigma) \leq a_\tau + \operatorname{codim}(\tau)$ if τ is a face of σ, which follows from a dimension count. □

Finally, we record for completeness the known first "gap" in the 3-dimensional spectrum predicted by the A.C.C. conjecture:

Proposition 2.5. *Assume (X, B) is a log variety of dimension 3 and let $x \in X$ be a closed point. Then the following hold:*

a) *$a(x; B) > 2$ iff X is nonsingular at x and $a(x; B) = 3 - \operatorname{mult}_x(B)$.*

b) $a(x;B)=2$ *iff one of the following holds:*
i) X *is nonsingular at* x *and* $\mathrm{mult}_x B=1$.
ii) $x\notin \mathrm{Supp}(B)$ *and* X *has a cDV singularity at* x *(cf.* [Rd80]*).*

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